

# A note on complex interpolation and Calderón product of quasi-Banach spaces

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**Abstract** In this paper, we prove that the inner complex interpolation of two quasi-Banach lattices coincides with the closure of their intersection in their Calderón product. This generalizes a classical result by Shestakov in 1974 for Banach lattices.

## 1 Introduction

In this paper we consider the relation between complex interpolations and Calderón products for quasi-Banach lattices. We begin with the definition of complex interpolation for quasi-Banach spaces (see, for example, [1, 4, 5]). Consider a *couple of quasi-Banach spaces*  $X_0, X_1$ , which are continuously embedding into a large topological vector space  $Y$ . The *space*  $X_0 + X_1$  is defined as

$$X_0 + X_1 := \{h \in Y : \exists h_i \in X_i, i \in \{0, 1\}, \text{ such that } h = h_0 + h_1\},$$

with

$$\|h\|_{X_0+X_1} := \inf\{\|h_0\|_{X_0} + \|h_1\|_{X_1} : h = h_0 + h_1, h_0 \in X_0 \text{ and } h_1 \in X_1\}.$$

Let  $U := \{z \in \mathbb{C} : 0 < \Re z < 1\}$  and  $\overline{U} := \{z \in \mathbb{C} : 0 \leq \Re z \leq 1\}$ . A map  $f: U \rightarrow X$  is said to be *analytic* if, for any given  $z_0 \in U$ , there exists  $\eta \in (0, \infty)$  such that  $f(z) = \sum_{j=0}^{\infty} h_n(z-z_0)^n$ ,  $h_n \in X$ , is uniformly convergent for  $|z-z_0| < \eta$ . A quasi-Banach space  $X$  is said to be *analytically convex* if there exists a positive constant  $C$  such that, for any analytic function  $f: U \rightarrow X$  which is continuous on the closed strip  $\overline{U}$ ,

$$\max_{z \in U} \|f(z)\|_X \leq C \max_{\Re z \in \{0,1\}} \|f(z)\|_X.$$

Suppose that  $X_0 + X_1$  is analytically convex. The *set*  $\mathcal{F} := \mathcal{F}(X_0, X_1)$  is defined to be the set of all functions  $f: U \rightarrow X_0 + X_1$  satisfying that

- (i)  $f$  is analytic and *bounded* in  $X_0 + X_1$ , which means that  $f(U) := \{f(z) : z \in U\}$  is a bounded set of  $X_0 + X_1$ ;

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- (ii)  $f$  is extended continuously to the closure  $\overline{U}$  of the strip  $U$  such that the traces  $t \mapsto f(j + it)$  are bounded continuous functions into  $X_j$ ,  $j \in \{0, 1\}$ ,  $t \in \mathbb{R}$ .

We endow  $\mathcal{F}$  with the *quasi-norm*

$$\|f\|_{\mathcal{F}} := \max \left\{ \sup_{t \in \mathbb{R}} \|f(it)\|_{X_0}, \sup_{t \in \mathbb{R}} \|f(1 + it)\|_{X_1} \right\}.$$

Let  $\mathcal{F}_0 := \mathcal{F}_0(X_0, X_1)$  be closure of all functions  $f \in \mathcal{F}$  such that  $f(z) \in X_0 \cap X_1$  for all  $z \in U$ . We now recall the definition of complex interpolations.

**Definition 1.1.** Let  $X_0, X_1$  be two quasi-Banach spaces such that  $X_0 + X_1$  is analytically convex. Then the *outer complex interpolation space*  $[X_0, X_1]_{\theta}$  with  $\theta \in (0, 1)$  is defined by

$$[X_0, X_1]_{\theta} := \{g \in X_0 + X_1 : \exists f \in \mathcal{F} \text{ such that } f(\theta) = g\}$$

and its *norm* given by  $\|g\|_{[X_0, X_1]_{\theta}} := \inf_{f \in \mathcal{F}} \{\|f\|_{\mathcal{F}} : f(\theta) = g\}$ . The *inner complex interpolation space*  $[X_0, X_1]_{\theta}^i$  with  $\theta \in (0, 1)$  is defined via the same as  $[X_0, X_1]_{\theta}$  with  $\mathcal{F}$  replaced by  $\mathcal{F}_0$ .

It easily follows from the definition that  $[X_0, X_1]_{\theta}^i \hookrightarrow [X_0, X_1]_{\theta}$  and  $X_0 \cap X_1$  is dense in  $[X_0, X_1]_{\theta}^i$ . If  $X_0$  and  $X_1$  are both Banach spaces, then it is known that the inner and outer complex methods coincide (see [1, 5]). For the general quasi-Banach cases, Kalton, Mayboroda and Mitrea [5] pointed out that the inner and the outer complex methods yield the same space if  $X_0$  and  $X_1$  are separable analytically convex quasi-Banach spaces. However, for quasi-Banach spaces without the separability condition, whether these two methods still coincide is still unclear (see [5]).

Let  $(\Omega, \mu)$  be a  $\sigma$ -finite measure space and  $L_0$  be the collection of all complex-valued  $\mu$ -measurable functions on  $\Omega$ . A quasi-Banach function space  $X$  on  $\Omega$  is called a *quasi-Banach lattice* if for every  $f \in X$  and  $g \in L_0$  with  $|g(x)| \leq |f(x)|$  for  $\mu$ -a.e.  $x \in \Omega$ , one has  $g \in X$  and  $\|g\|_X \leq \|f\|_X$ .

**Definition 1.2.** Let  $X_j \subset L_0$ ,  $j \in \{0, 1\}$ , be quasi-Banach lattices on  $(\Omega, \mu)$  and  $\theta \in (0, 1)$ . Then the *Calderón product*  $X_0^{1-\theta} X_1^{\theta}$  of  $X_0$  and  $X_1$  is the collection of all functions  $f \in L_0$  such that

$$\|f\|_{X_0^{1-\theta} X_1^{\theta}} := \inf \left\{ \|f_0\|_{X_0}^{1-\theta} \|f_1\|_{X_1}^{\theta} : |f| \leq |f_0|^{1-\theta} |f_1|^{\theta} \quad \mu\text{-a.e.}, \quad f_j \in X_j, j \in \{0, 1\} \right\}$$

is finite.

The first result concerning the relation between complex interpolations and Calderón products is due to Calderón [1]. He showed that if  $X_0$  and  $X_1$  are Banach lattices, then  $[X_0, X_1]_{\theta} \hookrightarrow X_0^{1-\theta} X_1^{\theta}$ . Later, Shestakov [10] (see also [9, 7]) in 1974 proved that the complex interpolation of two Banach lattices  $X_0$  and  $X_1$  is just the closure of their intersection  $X_0 \cap X_1$  in their Calderón product, namely,  $[X_0, X_1]_{\theta} = \overline{X_0 \cap X_1}^{\|\cdot\|_{X_0^{1-\theta} X_1^{\theta}}}$ . In 1998, Kalton and Mitrea [4] considered more general quasi-Banach cases. Indeed, they proved in [4, Theorem 3.4] that, *if  $X_0$  and  $X_1$  are analytically convex separable*

quasi-Banach lattices, then  $X_0 + X_1$  is also analytically convex and  $[X_0, X_1]_\theta = X_0^{1-\theta} X_1^\theta$ . The proof of this result was noticed later in [5] to be also feasible for the coincidence  $[X_0, X_1]_\theta^i = X_0^{1-\theta} X_1^\theta$ , and so in this case,

$$[X_0, X_1]_\theta = [X_0, X_1]_\theta^i = X_0^{1-\theta} X_1^\theta = \overline{X_0 \cap X_1}^{\|\cdot\|_{X_0^{1-\theta} X_1^\theta}}.$$

Notice that in Kalton and Mitrea's result [4, Theorem 3.4], there is a condition on the separability of the spaces  $X_0$  and  $X_1$ . An interesting question is, how is the relation between complex interpolations and Calderón products of quasi-Banach lattices which are not separable? Is Shestakov's result for Banach spaces also true for general quasi-Banach cases? In this note we give a positive answer for the inner complex interpolation.

**Theorem 1.3.** *Let  $\Omega$  be a Polish space,  $\mu$  a  $\sigma$ -finite Borel measure on  $\Omega$  and  $(X_0, X_1)$  a pair of quasi-Banach lattices on  $(\Omega, \mu)$ . If both  $X_0$  and  $X_1$  are analytically convex, then*

$$[X_0, X_1]_\theta^i = \overline{X_0 \cap X_1}^{\|\cdot\|_{X_0^{1-\theta} X_1^\theta}}, \quad \theta \in (0, 1).$$

To prove Theorem 1.3, we use another interpolation method, the Gagliardo-Peetre interpolation introduced by Peetre [8]. In 1985 Nilsson [7] proved a general result concerning the relation between Gagliardo-Peetre interpolation and Calderón product, which is a key tool used in this paper. This proof is different from the one used by Shestakov [10] for Banach lattices.

Throughout the paper, the *symbol*  $C$  denotes a positive constant which may vary from line to line. The *meaning* of  $A \lesssim B$  is given by: there exists a positive constant  $C$  such that  $A \leq CB$ . The *symbol*  $A \sim B$  means  $A \lesssim B \lesssim A$ .

## 2 Proof of Theorem 1.3

Let  $X$  be a quasi-Banach lattice and  $p \in [1, \infty]$ . The *p-converexification* of  $X$ , denoted by  $X^{(p)}$ , is defined as follows:  $f \in X^{(p)}$  if and only if  $|f|^p \in X$ . For all  $f \in X^{(p)}$ , define  $\|f\|_{X^{(p)}} := \| |f|^p \|_X^{1/p}$ . The lattice  $X$  is called *1/p-converex* if  $X^{(p)}$  is a Banach space. Moreover, a quasi-Banach lattice  $X$  is said to be of *type  $\mathfrak{E}$* , if there exists an equivalent quasi-norm  $\|\cdot\|_X$  such that  $(X, \|\cdot\|_X)$  is 1/p-converex for some  $p \in [1, \infty)$ ; see [7].

An important tool we used is the following equivalent characterization of analytically convex quasi-Banach lattice; see, for example, [3, 5]. In what follows,  $K_X$  denotes the *modulus of concavity* of a quasi-Banach space  $X$ , i.e., the smallest positive constant satisfying

$$\|x + y\|_X \leq K_X(\|x\|_X + \|y\|_X), \quad x, y \in X.$$

**Proposition 2.1.** *Let  $X$  be a quasi-Banach lattice. Then the following assertions are equivalent:*

- (i)  $X$  is analytically convex;
- (ii) there exists  $r > 0$  such that  $X$  is  $r$ -convex, namely,  $X^{(1/r)}$  is a Banach space;
- (iii)  $X$  is  $r$ -convex for all  $0 < r < (1 + \log_2 K_X)^{-1}$ .

It follows from Proposition 2.1 that all analytically convex quasi-Banach lattices are of type  $\mathfrak{E}$ .

We now prove one direction of Theorem 1.3 in the following theorem, which can be proved by an argument similar to that used for [4, Theorem 3.4]. For the sake of convenience, we give some details here.

**Theorem 2.2.** *Let  $\Omega$  be a Polish space,  $\mu$  a  $\sigma$ -finite Borel measure on  $\Omega$  and  $(X_0, X_1)$  a pair of quasi-Banach lattices of functions on  $(\Omega, \mu)$ . If both  $X_0$  and  $X_1$  are analytically convex, then*

$$[X_0, X_1]_\theta^i \hookrightarrow \overline{X_0 \cap X_1}^{\|\cdot\|_{X_0^{1-\theta} X_1^\theta}}, \quad \theta \in (0, 1).$$

*Proof.* Since the lattices  $X_0, X_1$  are analytically convex, by Proposition 2.1, we know that there exists  $r \in (0, 1]$  such that  $X_0, X_1$  are both  $r$ -convex lattices. By [4, Theorem 3.4] and its proof,  $X_0 + X_1$  is also  $r$ -convex and hence  $(X_0 + X_1)^{(1/r)}$  is a Banach space.

Since  $X_0 \cap X_1$  is dense in  $[X_0, X_1]_\theta^i$ , it suffices to prove

$$(X_0 \cap X_1, \|\cdot\|_{[X_0, X_1]_\theta^i}) \hookrightarrow \overline{X_0 \cap X_1}^{\|\cdot\|_{X_0^{1-\theta} X_1^\theta}}.$$

Let  $f \in X_0 \cap X_1$ . Then for any  $\varepsilon > 0$  there exists  $F \in \mathcal{F}_0(X_0, X_1)$  such that  $F(\theta) = f$  and  $\|F\|_{\mathcal{F}(X_0, X_1)} \leq \|f\|_{[X_0, X_1]_\theta^i} + \varepsilon$ . Since  $F$  is analytic in  $U$  and continuous in  $\overline{U}$ , for any  $z_0 \in U$ , there exist  $R > 0$  and  $f_k \in X_0 + X_1$  such that  $F(z) = \sum_{k \in \mathbb{N}_0} f_k(z - z_0)^k$  with uniformly convergent in  $X_0 + X_1$  for all  $|z - z_0| < R$ . Moreover, due to the Cauchy-Hadamard theorem, it holds  $\limsup_{k \rightarrow \infty} \|f_k\|_{X_0 + X_1}^{1/k} \leq R^{-1}$ . We also know that for  $\mu$ -almost every  $w \in \Omega$ , any  $\rho < R$  and  $q \leq r$ ,

$$|F(z_0)(w)|^q \leq \frac{1}{2\pi} \int_0^{2\pi} |F(z_0 + \rho e^{it})(w)|^q dt.$$

Since  $z \mapsto |F(z)|^q$  is continuous into  $(X_0 + X_1)^{(1/q)}$ , we know that, for any positive continuous functional  $\phi \in ((X_0 + X_1)^{(1/q)})^*$ ,

$$\phi(|F(z_0)|^q) \leq \frac{1}{2\pi} \phi \left( \int_0^{2\pi} |F(z_0 + \rho e^{it})|^q dt \right) \leq \frac{1}{2\pi} \int_0^{2\pi} \phi(|F(z_0 + \rho e^{it})|^q) dt,$$

hence  $z \mapsto \phi(|F(z)|^q)$  is subharmonic on  $U$ . Then

$$(2.1) \quad \phi(|F(\theta)|^q) \leq \int_{\mathbb{R}} P_0(\theta, t) \phi(|F(it)|^q) dt + \int_{\mathbb{R}} P_1(\theta, t) \phi(|F(1+it)|^q) dt,$$

where  $P_0$  and  $P_1$  are the components of the Poisson kernel on  $U$  satisfying  $\int_{\mathbb{R}} P_0(\theta, t) dt = 1 - \theta$  and  $\int_{\mathbb{R}} P_1(\theta, t) dt = \theta$ . Let  $f_0 := ((1 - \theta)^{-1} \int_{\mathbb{R}} P_0(\theta, t) |F(it)|^r dt)^{1/r}$  and  $f_1 := ((1 - \theta)^{-1} \int_{\mathbb{R}} P_1(\theta, t) |F(1+it)|^r dt)^{1/r}$ . It follows from the  $r$ -convexity of  $X_0$  and  $X_1$  that  $f_j \in X_j$  with  $\|f_j\|_{X_j} \leq \|F\|_{\mathcal{F}(X_0, X_1)}$ ,  $j \in \{0, 1\}$ . By (2.1),  $q \leq r$  and the positivity of  $\phi$ , we have  $|F(\theta)|^q \leq (1 - \theta)f_0^q + \theta f_1^q$ . Taking  $\log$  in both side and letting  $q \rightarrow 0$  then gives  $|f| = |F(\theta)| \leq f_0^{1-\theta} f_1^\theta$ . Thus,  $\|f\|_{X_0^{1-\theta} X_1^\theta} \leq \|F\|_{\mathcal{F}(X_0, X_1)} \lesssim \|f\|_{[X_0, X_1]_\theta^i}$ , as desired.  $\square$

To show the other direction, we need the following Gagliardo-Peetre interpolation method, which was introduced by Peetre [8].

**Definition 2.3.** Let  $X_0$  and  $X_1$  be a pair quasi-Banach spaces and  $\theta \in (0, 1)$ . We say  $a \in \langle X_0, X_1 \rangle_\theta$  if there exists a sequence  $\{a_i\}_{i \in \mathbb{Z}} \subset X_0 \cap X_1$  such that  $a = \sum_{i \in \mathbb{Z}} a_i$  with convergence in  $X_0 + X_1$  and for any bounded sequence  $\{\varepsilon_i\}_{i \in \mathbb{Z}} \subset \mathbb{C}$ ,  $\sum_{i \in \mathbb{Z}} \varepsilon_i 2^{i(j-\theta)} a_i$  converges in  $X_j$ ,  $j \in \{0, 1\}$ . We further require that

$$\left\| \sum_{i \in \mathbb{Z}} \varepsilon_i 2^{i(j-\theta)} a_i \right\|_{X_j} \leq C \sup_{i \in \mathbb{Z}} |\varepsilon_i|, \quad j \in \{0, 1\},$$

for some constant  $C$ . As a quasi-norm of  $\langle X_0, X_1 \rangle_\theta$ , we use  $\|a\|_{\langle X_0, X_1 \rangle_\theta} := \inf C$

Applying Proposition 2.1 and [7, Theorem 2.1] (see [7, (2.1)]), we have the following conclusion.

**Theorem 2.4.** Let  $\Omega$  be a Polish space,  $\mu$  a  $\sigma$ -finite Borel measure on  $\Omega$  and  $(X_0, X_1)$  a pair of quasi-Banach lattices of functions on  $(\Omega, \mu)$ . If both  $X_0$  and  $X_1$  are analytically convex, then

$$\langle X_0, X_1 \rangle_\theta = \overline{X_0 \cap X_1}^{\|\cdot\|_{X_0^{1-\theta} X_1^\theta}}, \quad \theta \in (0, 1),$$

and  $\|\cdot\|_{\langle X_0, X_1 \rangle_\theta}$  is equivalent to  $\|\cdot\|_{X_0^{1-\theta} X_1^\theta}$ .

From this conclusion, we deduce that  $X_0 \cap X_1$  is dense in  $\langle X_0, X_1 \rangle_\theta$  and, to prove  $\overline{X_0 \cap X_1}^{\|\cdot\|_{X_0^{1-\theta} X_1^\theta}} \hookrightarrow [X_0, X_1]_\theta^i$ , it suffices to show  $\langle X_0, X_1 \rangle_\theta \hookrightarrow [X_0, X_1]_\theta^i$ .

**Theorem 2.5.** Let  $\Omega$  be a Polish space,  $\mu$  a  $\sigma$ -finite Borel measure on  $\Omega$  and  $(X_0, X_1)$  a pair of quasi-Banach lattices of functions on  $(\Omega, \mu)$ . If both  $X_0$  and  $X_1$  are analytically convex, then

$$\langle X_0, X_1 \rangle_\theta \hookrightarrow [X_0, X_1]_\theta^i, \quad \theta \in (0, 1).$$

*Proof.* Let  $D(X_0, X_1, \theta)$  be the subspace of  $\langle X_0, X_1 \rangle_\theta$  consisting of all  $f \in \langle X_0, X_1 \rangle_\theta$  such that there exists a finite set  $E \subset \mathbb{Z}$  and  $\{f_k\}_{k \in E} \subset X_0 \cap X_1$  such that  $f = \sum_{k \in E} f_k$  in  $X_0 + X_1$ , and for any bounded sequence  $\{\varepsilon_k\}_{k \in E}$  of complex numbers  $\sum_{k \in E} \varepsilon_k 2^{k(j-\theta)} f_k$  converges in  $X_j$ , with

$$\left\| \sum_{k \in E} \varepsilon_k 2^{k(j-\theta)} f_k \right\|_{X_j} \lesssim \|f\|_{\langle X_0, X_1 \rangle_\theta} \sup_{k \in E} |\varepsilon_k|, \quad j \in \{0, 1\}.$$

Obviously,  $X_0 \cap X_1 \subset D(X_0, X_1, \theta)$ , and hence  $D(X_0, X_1, \theta)$  is dense in  $\langle X_0, X_1 \rangle_\theta$ . To complete the proof, it suffices to show

$$(D(X_0, X_1, \theta), \|\cdot\|_{\langle X_0, X_1 \rangle_\theta}) \hookrightarrow [X_0, X_1]_\theta^i.$$

Let  $f \in D(X_0, X_1, \theta)$ . Without loss of generality, we may assume that  $f = \sum_{|k| \leq M} f_k$  in  $X_0 + X_1$  for some  $M \in \mathbb{N}$  and  $\{f_k\}_{|k| \leq M} \subset X_0 \cap X_1$ , and

$$(2.2) \quad \left\| \sum_{|k| \leq M} \varepsilon_k 2^{k(j-\theta)} f_k \right\|_{X_j} \lesssim \|f\|_{\langle X_0, X_1 \rangle_\theta} \sup_{k \in E} |\varepsilon_k|, \quad j \in \{0, 1\}.$$

Define  $F(z) := \sum_{|k| \leq M} 2^{k(z-\theta)} f_k$  with convergence in  $X_0 + X_1$  for all  $z \in \overline{U}$ . Obviously,  $F(\theta) = f$  and  $F(z) \in X_0 \cap X_1$ .

Now we prove  $F \in \mathcal{F}_0(X_0, X_1)$ . The analyticity of  $F$  is obvious. To show  $F$  is bounded in  $X_0 + X_1$ , for  $z \in \overline{U}$ , write  $z = a + ib$  with  $a \in [0, 1]$  and  $b \in \mathbb{R}$ , and

$$F(z) = \sum_{-M \leq k < 0} 2^{ka+kb i} 2^{-k\theta} f_k + \sum_{0 \leq k \leq M} 2^{k(a-1)+kb i} 2^{k(1-\theta)} f_k =: F_0(z) + F_1(z).$$

Since  $\{2^{ka+kb i}\}_{-M \leq k < 0}$  and  $\{2^{k(a-1)+kb i}\}_{0 \leq k \leq M}$  are bounded sequences, by (2.2), we have

$$\|F_j(z)\|_{X_j} \lesssim \|f\|_{\langle X_0, X_1 \rangle_\theta}, \quad j \in \{0, 1\}.$$

This implies  $F(z) \in X_0 + X_1$  and  $\|F(z)\|_{X_0+X_1} \lesssim \|f\|_{\langle X_0, X_1 \rangle_\theta}$  for all  $z \in \overline{U}$ .

Similarly, since  $\{2^{kti}\}_{|k| \leq M}$  is a bounded sequence, applying (2.2) we obtain

$$\begin{aligned} \|F(j+it)\|_{X_j} &= \left\| \sum_{|k| \leq M} 2^{kit} 2^{k(j-\theta)} f_k \right\|_{X_j} \\ &\lesssim \|f\|_{\langle X_0, X_1 \rangle_\theta} \sup_{|k| \leq M} |2^{kti}| \lesssim \|f\|_{\langle X_0, X_1 \rangle_\theta}, \quad j \in \{0, 1\}. \end{aligned}$$

Now we show  $t \mapsto F(j+it)$  is a continuous function into  $X_j$ ,  $j \in \{0, 1\}$ . Fix  $t_0 \in \mathbb{R}$ . Notice that, for any  $\varepsilon > 0$ , we can find  $\delta = \delta(M, \varepsilon) > 0$ , such that for any  $|t - t_0| < \delta$  and  $|k| \leq M$ ,  $|2^{kit} - 2^{kit_0}| < \varepsilon$ . Hence,

$$\begin{aligned} \|F(j+it) - F(j+it_0)\|_{X_j} &= \left\| \sum_{|k| \leq M} [2^{kit} - 2^{kit_0}] 2^{k(j-\theta)} f_k \right\|_{X_j} \\ &\lesssim \|f\|_{\langle X_0, X_1 \rangle_\theta} \sup_{|k| \leq M} |2^{kit} - 2^{kit_0}| \lesssim \varepsilon \|f\|_{\langle X_0, X_1 \rangle_\theta}, \quad j \in \{0, 1\}, \end{aligned}$$

as desired.

It remains to show the extension of  $F$  from  $U$  to  $\overline{U}$  is continuous. Since  $F$  is analytic in  $U$ , we only need to prove that, for any  $t \in \mathbb{R}$ ,

$$(2.3) \quad \|F(a+it) - F(it)\|_{X_0+X_1} \rightarrow 0, \quad a \rightarrow 0^+$$

and

$$(2.4) \quad \|F(a+it) - F(1+it)\|_{X_0+X_1} \rightarrow 0, \quad a \rightarrow 1^-.$$

For any  $\varepsilon > 0$ , we can find  $\delta = \delta(M, \varepsilon) > 0$ , such that for any  $0 < a < \delta$  and  $|k| \leq M$ ,  $|2^{ka} - 1| < \varepsilon$ . Write

$$F(a + it) - F(it) = \sum_{-M \leq k < 0} [2^{ka} - 1] 2^{kit} 2^{-k\theta} f_k + \sum_{|k| \leq M} [2^{ka} - 1] 2^{-k} 2^{kit} 2^{k(1-\theta)} f_k.$$

Since

$$\left\| \sum_{-M \leq k < 0} [2^{ka} - 1] 2^{kit} 2^{-k\theta} f_k \right\|_{X_0} \lesssim \varepsilon \|f\|_{\langle X_0, X_1 \rangle_\theta}$$

and

$$\left\| \sum_{0 \leq k \leq M} [2^{ka} - 1] 2^{-k} 2^{kit} 2^{k(1-\theta)} f_k \right\|_{X_1} \lesssim \varepsilon \|f\|_{\langle X_0, X_1 \rangle_\theta},$$

we know that

$$\|F(a + it) - F(it)\|_{X_0 + X_1} \lesssim \varepsilon \|f\|_{\langle X_0, X_1 \rangle_\theta}.$$

This gives (2.3). A similar argument gives (2.4).

Combining the above arguments, we know that  $F \in \mathcal{F}_0(X_0, X_1)$  with  $\|F\|_{\mathcal{F}(X_0, X_1)} \lesssim \|f\|_{\langle X_0, X_1 \rangle_\theta}$ . Therefore,  $\|f\|_{[X_0, X_1]_\theta^i} \lesssim \|f\|_{\langle X_0, X_1 \rangle_\theta}$ . This finishes the proof.  $\square$

Theorem 1.3 is then a consequence of the above three theorems. Moreover, as a byproduct, we obtain the coincidence between the inner complex interpolation and the Gagliardo-Peetre interpolation.

**Corollary 2.6.** *Let  $(X_0, X_1)$  a pair of analytically convex quasi-Banach lattices of functions on  $(\Omega, \mu)$ . Then  $\langle X_0, X_1 \rangle_\theta = [X_0, X_1]_\theta^i$  for all  $\theta \in (0, 1)$ .*

**Remark 2.7.** Recall that if  $X_0$  and  $X_1$  are Banach spaces, then  $\langle X_0, X_1 \rangle_\theta \hookrightarrow [X_0, X_1]_\theta$ ; see, for example, [8, 2, 7]. The above corollary gives a generalization of this coincidence.

From the relation between the inner and outer complex interpolations (see [1] and [5, Theorem 7.9]), we also have the following conclusion.

**Corollary 2.8.** *Let  $\theta \in (0, 1)$  and  $(X_0, X_1)$  a pair of analytically convex quasi-Banach lattices of functions on  $(\Omega, \mu)$ . Then  $\langle X_0, X_1 \rangle_\theta = [X_0, X_1]_\theta$  if either  $X_0, X_1$  are both Banach spaces or  $X_0, X_1$  are both separable.*

Finally we give an application of Theorem 1.3 to the Morrey space, which is a typical example of non-separable spaces. Let  $0 < p \leq u \leq \infty$  and  $(\mathcal{X}, \mu)$  be a quasi-metric measure space. Recall that the Morrey space  $\mathcal{M}_p^u(\mathcal{X})$  is the collection of all  $p$ -locally integrable functions  $f$  on  $\mathcal{X}$  such that

$$\|f\|_{\mathcal{M}_p^u(\mathcal{X})} := \sup_{B \subset \mathcal{X}} |B|^{1/u-1/p} \left[ \int_B |f(x)|^p dx \right]^{1/p} < \infty,$$

where the supremum is taken over all balls  $B$  in  $\mathcal{X}$ . Obviously,  $\mathcal{M}_p^p(\mathcal{X}) = L_p(\mathcal{X})$ . Since Morrey spaces are non-separable, we can not apply [4, Theorem 3.4] to Morrey spaces.

By [6, Proposition 2.1], we know that  $[\mathcal{M}_{p_0}^{u_0}(\mathcal{X})]^{1-\theta} [\mathcal{M}_{p_1}^{u_1}(\mathcal{X})]^\theta = \mathcal{M}_p^u(\mathcal{X})$ , which together with Theorem 1.3 induce the following conclusion.

**Proposition 2.9.** *Let  $\theta \in (0, 1)$ ,  $0 < p_i \leq u_i < \infty$ ,  $i \in \{0, 1\}$  and*

$$\frac{1}{u} := \frac{1-\theta}{u_0} + \frac{\theta}{u_1}, \quad \frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

*If  $u_0 p_1 = u_1 p_0$ , then*

$$[\mathcal{M}_{p_0}^{u_0}(\mathcal{X}), \mathcal{M}_{p_1}^{u_1}(\mathcal{X})]_{\theta}^i = \langle \mathcal{M}_{p_0}^{u_0}(\mathcal{X}), \mathcal{M}_{p_1}^{u_1}(\mathcal{X}) \rangle_{\theta} = \overline{\mathcal{M}_{p_0}^{u_0}(\mathcal{X}) \cap \mathcal{M}_{p_1}^{u_1}(\mathcal{X})}^{\|\cdot\|_{\mathcal{M}_p^u(\mathcal{X})}}.$$

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